CAMBRIDGE
International Education

Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/43**

Paper 4 Further Probability & Statistics **May/June 2024**

MARK SCHEME

Maximum Mark: 50

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2024 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

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Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptions for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles

1 Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.

2 Unless specified in the question, non-integer answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.

3 Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.

4 Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).

5 Where a candidate has misread a number or sign in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 A or B mark for the misread.

6 Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

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Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

<table>
<tr><th>Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td rowspan="4">1</td><td>
<table>
<tr><td>21.4</td><td>4</td><td>20.9</td><td>2</td></tr>
<tr><td>24.6</td><td>15</td><td>23.5</td><td>11</td></tr>
<tr><td>25.3</td><td>18</td><td>24.8</td><td>16</td></tr>
<tr><td>22.7</td><td>8</td><td>21.9</td><td>6</td></tr>
<tr><td>20.8</td><td>1</td><td>23.4</td><td>10</td></tr>
<tr><td>21.5</td><td>5</td><td>24.0</td><td>13</td></tr>
<tr><td>22.9</td><td>9</td><td>23.8</td><td>12</td></tr>
<tr><td>21.3</td><td>3</td><td>24.1</td><td>14</td></tr>
<tr><td>22.3</td><td>7</td><td>25.1</td><td>17</td></tr>
<tr><td></td><td></td><td>25.8</td><td>19</td></tr>
</table>
</td><td>M1 A1</td><td>Attempt at ranking from 1 to 19.
A1 for all correct.</td></tr>
<tr><td>Test statistic: 70</td><td>A1</td><td>If ranks reversed, then sum of ranks = 110, and then 9(9 + 10 + 1) – 110 = 70.</td></tr>
<tr><td>H_0: population median of X = population median of Y
H_1: population median of X < population median of Y</td><td>B1</td><td>Accept H_0: $m_X = m_Y$ H_1: $m_X < m_Y$.
Do not accept any other notation.</td></tr>
<tr><td>Critical value for $m = 9$, $n = 10$ is 61</td><td>B1</td><td>Using normal approximation implies last 3 marks are B0M0A0, max 4 out of 7.</td></tr>
</table>

Question	Answer	Marks	Guidance
1	'70' > '61' accept H_0	**M1**	'70' must come from ranks 1 – 19, '61' must be a critical value from the correct table. ft conclusion for their 70 and their 61. Condone Reject H_1 . 'Accept H_0' can be implied by a conclusion in context that is consistent with their 70 and their 61.
	Insufficient evidence to suggest that population medians of *X* and *Y* are not equal or Insufficient evidence to suggest that population median of *X* is less than median of *Y* or Insufficient evidence to support the interviewer's belief	**A1**	Correct work only, ignoring their hypotheses, conclusion in context with level of uncertainty in language. Not 'prove' or equivalent. Condone 'no sufficient' 'not enough'. Do not accept statements such as 'there is sufficient evidence to suggest…..'.
		7	

Question	Answer	Marks	Guidance
2(a)	$\overline{x} = \frac{1}{2}(68.66 + 64.22) = 66.44$	**M1**	
	$\sum x = 12 \times 66.44 = 797.28$	**A1**	May come from use of incorrect *t* value. Accept AWRT 797.
	$\frac{ts}{\sqrt{12}} = \frac{1}{2}(68.66 - 64.22)$	**M1**	Must be with a *t* value (not *z* value).
	With $t = 2.718$ gives $s^2 = 8.0055$	**A1**	Accept AWRT 8.01; may be recovered by final answer exactly 53059(.34). $s = 2.8294$.
	Use $s^2 = \frac{1}{11}\left(\sum x^2 - \frac{(\sum x)^2}{12}\right)$ to find an expression for $\sum x^2$	**M1**	Must see an expression for $\sum x^2$.
	$\sum x^2 = 53059$	**A1**	AWRT 53100 but not from wrong working.
		6	
2(b)	Population is normally distributed	**B1**	Underlying distribution is normal B1. Pulse rates are normal B1. Population **mean** is normally distributed B0. (Underlying) **data** is normal B0.
		1	

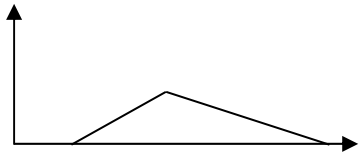
Question	Answer	Marks	Guidance
3	<table><tr><th></th><th>Bus company *A*</th><th>Bus company *B*</th><th>Bus company *C*</th></tr><tr><td>Arrived early</td><td>22 (**17.28**)</td><td>22 (**21.60**)</td><td>10 (**15.12**)</td></tr><tr><td>Arrived on time</td><td>30 (**39.68**)</td><td>52 (**49.60**)</td><td>42 (**34.72**)</td></tr><tr><td>Arrived late</td><td>28 (**23.04**)</td><td>26 (**28.80**)</td><td>18 (**20.16**)</td></tr></table>	**M1 A1**	Calculate expected values, at least 4 correct to 3sf. These must be seen. A1 for all correct to at least 3sf.
	<table><tr><td>1.2893</td><td>0.00741</td><td>1.7338</td></tr><tr><td>2.3615</td><td>0.1161</td><td>1.5265</td></tr><tr><td>1.0678</td><td>0.2722</td><td>0.2314</td></tr></table>	**M1**	At least 4 correct values (to 2sf) or expressions. May be implied by 8.600 – 8.615.
	Test statistic = 8.61	**A1**	8.600 – 8.615 Unsupported 8.600 – 8.615 scores M0A0 M1A1.
	H_0: Reliability of buses **is independent of** bus **company** **H_1: Reliability** of buses is **not independent of/dependent on** bus **company**	**B1**	Condone ‘No association/relationship between…’ ‘Association/relationship between…’
	‘8.61’ < 9.488 Accept H_0	**M1**	‘8.61’ from an attempt at chi-squared, 9.488 must be correct. ft conclusion from their 8.61. Condone ‘Reject H_1’. ‘Accept H_0’ can be implied by a conclusion in context that is consistent with their 8.61.

Question	Answer	Marks	Guidance
3	There is insufficient evidence to support/ suggest that reliability of buses is dependent/not independent of bus company	**A1**	Correct work only, ignoring their hypotheses, conclusion in context with level of uncertainty in language. Not 'prove' or equivalent. Condone 'no sufficient' 'not enough'. Do not accept statements such as 'there is sufficient evidence to suggest…..'
		7	

Question	Answer	Marks	Guidance
4(a)	$c = \frac{1}{32}$	**B1**	Accept 0.03125 or 0.0313.
		1	
4(b)	$G_X(t) = ct(1+t)^5$, $G'_X(t) = c\left(5t(1+t)^4 + (1+t)^5\right)$	**M1**	Must use product rule OR differentiate their expansion of G Allow one error in their differentiation. Condone missing *c*.
	$E(X) = G'_X(1) = c(80+32) = 3.5$	**A1**	Correct work only.
		2	

Question	Answer	Marks	Guidance
4(c)	$G_Y(t) = c^2t^2(1+t)^{10}$	***M1**	Or $c^2\left(t^2+10t^3+45t^4+120t^5+210t^6+252t^7+210t^8+120t^9+45t^{10}+10t^{11}+t^{12}\right)$
	$G'_Y(t) = c^2\left(10t^2(1+t)^9+2t(1+t)^{10}\right)$ $G''_Y(t) =$ $c^2\left(90t^2(1+t)^8+40t(1+t)^9\right)+2(1+t)^{10})$	**DM1**	Differentiate their $G_Y(t)$ twice, allow one error.
	$G'_Y(1) = 7$, $G''_Y(1) = 44.5$ $\text{Var}(Y) = G''_Y(1) + G'_Y(1) - \left(G'(1)\right)^2 = 2.5$	**M1 A1**	Use correct formula and substitute values for $t = 1$ (may be in terms of c). A1 for correct work only.
	Alternative solution for question 4(c)		
	$\text{Var}(Y) = 2\text{Var}(X)$	***M1**	
	Differentiate $G_X(t)$ twice	**DM1**	Note that $G'_X(t)$ has already been found in part **(b)**.
	$\text{Var}(X) = G''_X(1) + G'_X(1) - \left(G'_X(1)\right)^2$	**M1**	Use correct formula and substitute values for $t = 1$.
	$1.25 \times 2 = 2.5$	**A1**	
		4	

Question	Answer	Marks	Guidance
4(d)	P($Y = 5$) = coefficient of t^5 in expansion of their $c^2t^2(1+t)^{10}$ Required term is $c^2 \times 120$ Expansion = $c^2t^2(1+10t+45t^2+120t^3+\ldots)$	**M1**	Identify required term.
	$\frac{15}{128}$	**A1**	0.117, $\frac{120}{1024}$ or any equivalent fraction.
		2	

Question	Answer	Marks	Guidance
5(a)	$\frac{x-2}{6}$ $\frac{8-x}{12}$	**M1**	Attempt at differentiating both parts of CDF to obtain linear expressions. May be implied by first A1.
	Sketch 	**A1**	Correct shape (Two line segments touch the x-axis and meet at a point above the x-axis).
		A1	Correct shape with 2 and 8 and at least one of 4 or 1/3 correctly labelled.
		3	

Question	Answer	Marks	Guidance
5(b)	$\int_2^4 \frac{1}{6}x(x-2)\,\mathrm{d}x + \int_4^8 \frac{1}{12}x(8-x)\,\mathrm{d}x$	**B1 FT**	FT *their* PDF, with correct limits.
	$\frac{1}{6}\left[\frac{1}{3}x^3 - x^2\right] + \frac{1}{12}\left[4x^2 - \frac{1}{3}x^3\right]$	**M1**	Attempt at integration with *their* PDF (not CDF) no limits needed. May be implied by correct answer.
	$\frac{14}{3}$	**A1**	Accept 4.67 or any equivalent fraction.
		3	
5(c)	LQ: $\frac{1}{12}(x-2)^2 = \frac{1}{4}, \quad x = 2+\sqrt{3}$	**B1**	Allow 3.73.
	UQ: $1 - \frac{(8-x)^2}{24} = \frac{3}{4}, \quad x = 8-\sqrt{6}$	**B1**	Allow 5.55.
	IQR = UQ – LQ, with values found for both quartiles	**M1**	
	$6-\sqrt{6}-\sqrt{3}$	**A1**	Exact answer required.
		4	

Question	Answer	Marks	Guidance
6(a)	$s_x^2 = \frac{1}{49}\left(23480 - \frac{1080^2}{50}\right)$ (= 3.102) $s_y^2 = \frac{1}{39}\left(22220 - \frac{940^2}{40}\right)$ (= 3.333)	**M1**	$\frac{152}{9}$ $\frac{10}{3}$ Both.
	$s^2 = \frac{3.102}{50} + \frac{3.333}{40}$ $\left(= 0.1454\right)$	**M1**	
	$s^2 = \frac{2137}{14700} = 0.1454$	**A1**	
	CI: $\frac{1080}{50} - \frac{940}{40} \pm 1.645\sqrt{s^2}$ or $\frac{940}{40} - \frac{1080}{50} \pm 1.645\sqrt{s^2}$	**M1 B1**	Use of correct formula with a z value. B1 for 1.645 (accept 1.64 or 1.65) used in CI formula.
	$\left[1.27, 2.53\right]$ or [–2.53, –1.27]	**A1**	(1.27, 2.53) A1. $1.27 \leqslant \mu_Y - \mu_X \leqslant 2.53$ A1 (ignore symbol used inside the inequality). Condone [2.53, 1.27] or [–1.27, –2.53] A1. Final answer 1.9 ± 0.627 A0. Pooled estimate: max score M1 M0A0 M1B1A0. First M1 for correct form of pooled estimate.
		6	

Question	Answer	Marks	Guidance
6(b)	$z = \frac{1.9 - 1.1}{s}$	**M1**	$z = \frac{-1.9 - 1.1}{s}$ or -7.87 implies M1.
	2.098	**A1**	AWRT 2.10
	'2.098' > 1.28(2) Reject H_0 or $\Phi(2.098) = 0.9821 > 0.9$ Reject H_0 or $1 - \Phi(2.098) = 0.0179 < 0.1$ Reject H_0	**M1**	'2.098' from their attempt at z, 1.28(2) must be correct. Consistent signs in comparison. FT conclusion from their 2.098. Condone 'Accept H_1'. 'Reject H_0' can be implied by a conclusion in context that is consistent with their 2.098.
	Sufficient evidence to suggest that the **lengths** of tails of wallabies **in** region ***Y*** are, **on average**, **more than 1.1 cm longer than** the tails of wallabies **in** region ***X***	**A1**	Correct work only, conclusion in context with level of uncertainty in language. Not 'prove' or equivalent. Accept 'enough evidence'. Condone 'some evidence'.
		4	Note that it is possible to use part **(a)** but it must be made clear explicitly that rejection at 5% level implies rejection at 10% level.